

ON NEARLY S-PERUTABLE AND NEARLY HALL S-EMBEDDED SUBGROUPS: IMPLICATIONS FOR SUPERSOLVABILITY AND P-NILPOTENCY OF GROUPS

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ABSTRACT

A subgroup T of G is said to be nearly S -permutable if $(p, |T|) = 1$, for some prime p . Also, for every $H \leq G$ containing T the normalizer $N_k(T)$ contain some Sylow p -subgroups of H . Furthermore, T is called nearly Hall S -embedded if for some normal subgroup N of G s.t $TN (\leq G)$ is a Hall and $T \cap N \leq T \leq T_s(G)$, where $T_s G$ is the largest nearly S -embedded of G contained in T and T is nearly S -permutable provided supplement $B \leq T$ such that T permutes with every Sylow P of B . Now we will show supersolvability and p -nilpotency of some groups.

Keywords: Nearly S -permutable, supersolvability, p -nilpotent, Hall subgroups

MSC (2020): 20D10, 20D15, 20D20.

INTRODUCTION

Only finite groups will be discovered in this article. All the symbols and notations are standard. A finite group G and its sub groups denoted by H and T . Sylow subgroups of a finite group are denoted by $Syl_p(G)$. A permutable subgroup S is denoted as $SH = HS, SH \leq G$. There are interesting results and generalizations related to S -permutable are shown in [3-6]. As subgroup T of G is called S -permutable if $TQ = QT \forall Q \in Syl_p(G)$ (see [1]) S -permutable subgroups are also known as S -quasinormal and the subgroups consist of very interesting series of results. For example; If a subgroup T is S -permutable than this subgroup is subnormal in G [1]. Also, from [2] the subgroup $T H / H_G$ is nilpotent. Now we will present the basic definitions of this paper.

Definition: 1.1

Suppose $T(\leq G)$ and is known as nearly S -permutable $\forall P$ with $(P, |T|) = 1$ and $\forall TH \leq G$ contains T . The symbol $N_h(T)$ is normalizer containing $Syl_p(H)$.

Definition: 1.2

Subgroups $T \leq G$ is called nearly Hall S -embedded in G if for any $N \leq G$ satisfying TN a Hall subgroup of G $\& T \cap N \leq T \leq T_s(G)$ where $T_s G$ is the largest nearly S -permutable contained in T .

The group G with maximal Sylow subgroups, NH -embedded in G are supersolvable [8]

From nearly Hall S -embedded and nearly S -permutable subgroup it is clear that normal subgroups and Hall subgroups are nearly Hall S -embedded subgroups. On the other hand, converse does not hold. In this article, we will present some

generalizations related to supersolvability and p -nilpotency and the following theorem will be proved.

Theorem: 1.3

If every maximal subgroup of every $Syl_p(G)$ is either nearly S -permutable or nearly Hall S -embedded, then G is supersolvable.

Theorem: 1.4

Suppose that for any finite group G and $G_p \in Syl_p(G)$ for $|G|/p$ then p is the smallest number. Consider every maximal subgroup $H \leq G_p$, either nearly S -permutable or nearly Hall S -embedded subgroup. Then G is p -nilpotent.

Theorem: 1.5

Suppose $H \cong G$ (of a finite group) and G/H be a supersolvable and every maximal subgroup every maximal subgroup T of $Syl(H)$ is either nearly S -permutable or nearly Hall S -embedded. Then G is supersolvable.

Preliminaries: 2

Here we will prove our basic results to generalize the idea of nearly Hall S -embedded subgroups of finite groups.

Lemma: 2.1 [7]

Suppose a NS-permutable Subgroup $T \leq G$ and $M \triangleleft G$. Then

- (1) TM is NS-permutable.
- (2) If prime power order group T , then $T \cap M$ is NS-permutable in G .
- (3) If prime power order so T^M/M is NS-permutable in G/M for any $M \triangleleft G$.

Lemma: 2.2 [8]

$|T| = q^n$, Where q^n is a prime, so
 $T \leq Q_q(G)$

And T is NS-permutable in G .

Lemma: 2.3 [9]

Consider $T \leq G$ is S -permutable so T/T_G nilpotent, T_G is the core of $T \leq G$.

Lemma: 2.4 [10]

If $T \leq G$ is S -permutable and T is a prime group with prime number q

So,

$$O^q(G) \leq N_G(T)$$

Lemma: 2.5 [11]

Suppose $T \leq G$, where T is nilpotent so,

- (1) $T \leq G$ where T is S -permutable.
- (2) $Syl(H)$ is S -quasinormal.

Lemma: 2.6 [12]

Consider Q to be a $Syl_q(G)$ and $Q \circ \max(Q)$, then

- (1) $Q \circ \triangleleft G$.
- (2) $Q \circ$ is S -quasinormal.

Lemma: 2.7 [8]

Suppose $T \leq G$ and T is NH-embedded. Consider $M \triangleleft G$

- (1) If $T \leq H \leq G$ with H subnormal, T is NH-embedded in H .
- (2) Consider T a q -group for $q \in \pi(G)$ if $M \leq G$, then T/M is NH-embedded in G/M .
- (3) M is a q -subgroup, so for any q -subgroup T , T^M/M is NH-embedded of G/M

Lemma: 2.8 [7,12]

Suppose $T \leq H \leq K$

- (1) Consider $M \triangleleft K$ and T to be a prime number group. If T is nearly S -permutable, so TM/M is nearly S -permutable in G/M .
- (2) Consider T is nearly S -permutable contained in G and $T \leq O_p(G)$, then T is nearly S -permutable in G .

Lemma: 2.9 [13]

Consider T to be a nearly S -permutable subgroup in G , so T is a prime group for $p \in \pi(G)$ if $M \triangleleft G$, then $T \cap M$ is also Nearly S -permutable.

Lemma: 2.10 [14]

Suppose R, S, T contained in a finite group G . So equivalent statements are,

$$R \cap ST = (R \cap S)(R \cap T)$$

$$RS \cap RT = R(S \cap T)$$

3. Proof of theorems

Theorem 1.3

Proof:

We check it conversely, consider that this statement is not true and its example is G . Suppose q is the prime (smallest) which divides the order of G . So, G will be nilpotent we will prove it in the next theorem. Suppose R is a normal and Hall q '-subgroup contained in G . We can check that R follows the results of Lemma 2.1 and Lemma 2.7. R is supersolvable by induction and G consists of $S_y l$ Power property of supersolvable. Now consider P is the prime number (largest) which divides the order of G and P is $S_y l(G)$. So, $P \triangleleft G$, From Lemma 2.1 and Lemma 2.7 it is clear G/P possesses the hypothesis. Thus G/P is supersolvable.

Suppose $M \triangleleft G$ where M is a minimal normal subgroup. Consider it as G/M satisfied this, G/M is supersolvable. We know that all the supersolvable is saturated formation, $M \triangleleft G$ where M is normal and unique.

So M is the subgroup of P , now we claim $M \leq T$ $T \in N(P)$. From our supposition T is Nearly Hall S -embedded or simply SS -quasinormal.

Now consider if T SS -quasinormal in G .

$$Q \triangleleft G$$

Moreover, $T \triangleleft G$ from Lemma 2.6 thus $M \leq T$.

Consider T is nearly hall S -embedded contained in G . $T \triangleleft G$ such that TH be a hall subgroup and $T \cap H \leq H$. And $H = 1$, so $T = TH$ is Hall subgroup contained in G .

$$\Rightarrow T = 1 \text{ and so } |P| = P$$

As we know G/P supersolvable, we have $G \Rightarrow$ supersolvable. This contradiction $H \neq 1$ and $M \leq T$. In this way G/H is supersolvable, if $|T \cap H|$ equal to 1 then $|P \cap H|$ is p .

This conclude $M = P \cap H$ with p order, so G is supersolvable.

Again, if $T \cap H$ not equal to 1 check $T \cap H \leq T_{S \in G} \leq T$, and

$T \cap H = T_{S \in G} \cap H$ is nearly S -permutable using 2.9 Lemma. Next, we have

$$T \cap H \leq P = O_p(G) \Rightarrow T \cap H$$

Be S -quasinormal or S -permutable using 2.8 Lemma. Using Lemma 2.4 $O'(G) \leq M_G(T \cap H)$.

Note that $T \cap H \triangleleft G$.

$$\text{Hence } M \leq T \cap H \leq T$$

As required, we have

$$M \leq \cap T = \emptyset(P) \quad ([6]) \\ \emptyset(P) \leq \emptyset(G)$$

And this $M \leq \emptyset(G)$

$G/\emptyset(G)$ supersolvable. This conclude G Supersolvable which contradicts the statement and hence theorem proved.

Theorem 1.4

Proof:

We check conversely the statement is not true and its example is group G with minimal order. Suppose a $S_y l_q(G)$ and set of all maximal subgroups of $G_q \{Q_1, Q_2, Q_3, \dots, Q_n\}$. From the statement of the theorem every element Q_1 of $M(G_q)$ is either Nearly Hall S -embedded subgroup or nearly S -permutable without losing generality, Consider every element of $G_1(G_q) = G_2(G_q) = \{Q_{k+1}, Q_{k+2}, \dots, Q_n\}$ of $M(G_q)$ is nearly S -permutable for $1 \leq k \leq n$.

This prove can be solved in 5 states as below.

1) $M \triangleleft G$, where M is minimal and G/M is q -nilpotent. Consider $M \triangleleft G$ where M is minimal, so $G_q M/M$ is $S_y l_q(G/M)$ for only $N/M \in M(G_q M/M)$, suppose $Q = N \cap G_q$

So,

$$N = N \cap G_q M = (N \cap G_q) M = QM$$

and

$$Q \cap M = (N \cap G_q) \cap M = QM \cap G_q \cap M \\ = G_q \cap M$$

because

$$|G_q : Q| = |G_q(G_q \cap N)| = |G_q N : N| = Q$$

as

$Q \in M(G_p)$ From our supposition Q is either Nearly Hall s -embedded ∞ – Nearly s -permutable. Now let Q is Nearly s -permutable, so $N/M = QM/M$ is also Nearly s -permutable in G/M using Lemma 2.1.

Here we suppose Q is nearly Hall s -embedded contained in G , so $]H \triangleleft G$ s.t QH is a Hall subgroup in G and $Q \cap H \leq Q_{SG}$ clearly $HM/M \triangleleft G/M$, and

$$QM/M \cap HM/M = QHM/M$$

Is a Hall subgroup of G/M as $Q \cap M = G_q \cap M$ is $S_y l(M)$, we have

$$|M \cap QH|_{q'} = |M|_q = |M \cap Q|_q \\ = |(M \cap Q)(M \cap H)|_q$$

$$|M \cap QH|_{q'} = \frac{|QH|_{q'} |M|_q}{|MQH|_{q'}} = \frac{|H|_{q'} |M|_q}{|MH|_{q'}}$$

$$= |M \cap Q|_q |(M \cap Q)(M \cap H)|_q$$

$$M \cap QH = (M \cap Q)(M \cap H)$$

And thus

$$QM \cap HM = (Q \cap H)M$$

Using Lemma 2.11, so

$$\frac{QM}{M} \cap \frac{HM}{M} = \frac{\frac{QM}{M} \cap HM}{M} = \frac{(Q \cap H)M}{M}$$

$$\leq Q_{SG}M/M$$

As Q_{SG} is nearly S-permutable in G ,

So $Q_{SG}M/M$ is nearly S-permutable in G/M using Lemma 2.8

$$Q_{SG} \frac{M}{M} \leq (QM/M)_{SG} \frac{M}{M}$$

Thus, $N/M = QM/M$ is nearly Hall S-embedded in G/M .

From all these collections we can say that G/M satisfied the hypothesis. As the choice of G , it is obvious that G/M is p-nilpotent, also $M \triangleleft G$ and unique minimal. As p-nilpotent class is saturated.

(2) $Q_{q'}(G)$ is equal to 1.

Consider $Q_{q'}(G) \geq 1$, so $M \leq Q_{q'}(G)$ by (1)

by (1) since G/M is p-nilpotent and $G/Q_{q'}(G)$ is p-nilpotent, thus G is also p-nilpotent. This contradicts is our hypothesis. So $Q_{q'}(G) = 1$.

(3) $M \leq Q_i$ for any $Q_i \in M_i(G_q)$ for $T \in M_i(G_q)$, T is nearly Hall s-embedded in G then $T \triangleleft G$ such that TH is Hall subgroup and $T \cap H \leq T_{iG}$. If $H = 1$, so T is a Hall subgroup of G and thus $T = 1$ which implies $|G_q| = q$, as Q is maximal of G_q . By Burnside Theorem, G is p-nilpotent which is contradicts. Thus $H \neq 1$ and $M \leq H$, if $T \cap M = 1$, then $|M|_q \leq q$. Hence M is p-nilpotent using Burnside Theorem consider V be normal Hall q' -subgroup of M then $V \triangleleft G$. By minimality of $MV = 1$. Thus $|M| = q$ consequently, this nilpotency of $G/M \Rightarrow G$ is p-nilpotent which contradicts. So, we have $T \cap M \neq 1$ and $T \cap M \leq T \cap H \leq T_{SG} \leq T$.

We have $T_{SG} \cap M \leq T \cap M$ and thus $T \cap M = T_{SG} \cap M$. Using Lemma 2.9 $T \cap M$ is nearly S-permutable. M is solvable by Lemma 2.10

$\Rightarrow M$ is q' -group and $M \leq Q_q(G)$

Particularly $T \cap M \leq Q_q(G)$ by Lemma 2.8, $T \cap M$ is nearly s-permutable. Hence $O(G) \leq N_G(T \cap M)$ using Lemma 2.4 see $T \cap M \triangleleft G_q$ which implies $T \cap M \triangleleft G$.

Hence $T \cap M = M$ it leads to $M \leq T$.

(4) For $Q_j \in M_i(G_q)$, a normal N_j subgroup of G , s.t $M \leq N_j$. For any $T \in M_i(G_q)$ we have H is ss-quasinormal so $\exists B \leq G$ such that $G = TB$ and $TB_q = B_qT$ for every $Syl_{B_q}(B)$.

Thus $|B \cap T|_q = |G|_q = q$ from $G = TB$ so $B_q \notin T$ and $B_qT = B_qT$ is a $Syl_{B_q}(G)$. $H \in M_i(G_q)$ and comparing of orders $|T \cap B|_q = T \cap B$ thus

$$T \cap B = \bigcap_{b \in B} (B_q^b \cap T) \leq \bigcap_{b \in B} B_q^b = O_q(B)$$

From $|O_q(B)B \cap T| = q$ or 1, obtaining $|B/O_q(B)|_q = 1$ or q is the smallest prime dividing $|G|$ using Burnside Theorem.

$B/O_q(B)$ is nilpotent B being p-solvable and Hall q' -subgroup of B [1]. Let $H = \text{Hall } q' \text{-subgroup of } B, \pi$

$H = (P_2, \dots, P_5), P_i \in Syl_{P_i}(H)$ from ss-quasinormal subgroup definition, T and $\langle P_2, \dots, P_5 \rangle = K$ are permutations. So, TH is subgroup of G .

H is Hall q' -subgroup and TH is a subgroup of index q in G . q is smallest prime dividing $|G|, TH \triangleleft G$ if $TH = 1$.

Then G is elementary commutative q -group, which is contradicts. Hence

$$M \leq TH = N_j$$

T is supersolvable. Consider p is largest prime that divides order of T and $P = O_q(G)$ char $Syl_q(T)$, So $P \triangleleft T$ contained in G . Clearly $(G/P)/(T/P) \cong G/T$ supersolvable. Using Lemma 2.1 and 2.7 every max $Syl(T/P)$ is either Nearly Hall s-embedded or nearly S-permutable in G/P . Thus G/P endorse the statement and supersolvable.

(5) Final contradiction:

Take

$$V = \left(\bigcap_{j=1}^k Q_j \right) \cap \left(\bigcap_{i=k+1}^n N_i \right)$$

By the statement given above, we know $N \leq V$.

We have

$$\begin{aligned}
 M = M \cap G_q &\leq V \cap G_q \\
 &= \left(\left(\bigcap_{j=1}^k Q_j \right) \cap \left(\bigcap_{i=k+1}^n N_i \right) \right) \cap G_q \\
 &= \bigcap_{j=1}^n Q_j = \emptyset(G_q)
 \end{aligned}$$

From ([6]. III.3.3) we have $\emptyset(G_q) \leq \emptyset(G)$ and thus $M \leq \emptyset(G)$. As G/M is p-nilpotent we get $G/\emptyset(G)$ p-nilpotent. We know class of all p-nilpotent is saturated formation, G will be p-nilpotent. Which contradicts and proves the theorem.

Theorem 1.5

Proof:

Consider it false and take a minimal order example to prove. Using Lemma 2.1 and 2.7 we know every $\max S_{y,l}(T)$ either nearly Hall s-embedded or nearly s-permutable in H. By Theorem 1.3 consider $M \Delta G$ where minimal subgroup same as Theorem 1.4, we have G/M satisfies it and supersolvable. We know saturated formation consist of class of all supersolvable group, $M \Delta G$ will unique minimal contained in P.

Follow the proof of Theorem 1.3, we have $M \leq \emptyset(P)$, thus $G/\emptyset(P)$ supersolvable by ([6], iii, 3.3) $\emptyset(P) \leq \emptyset(G)$. Thus $G/\emptyset(G)$ supersolvable. As saturated formation G is supersolvable. If contradict thus complete the proof.

4. Applications

Corollary 4.1 ([5, Theorem 3.1]). Suppose q is the smallest prime dividing the order of G and $S_{y,l}(G)G_q$. Consider every maximal subgroup G_q is nearly Hall s-embedded in G. Thus, G is nilpotent.

Corollary 4.2 ([1, Theorem 3.1]). Suppose q is the smallest prime dividing order of G and $S_{y,l}(G)G_q$. Suppose every maximal G_q subgroup is nearly s-quasigroup subgroup in G. then G is nilpotent.

Corollary 4.3 ([5, Theorem 3.4]). Suppose a $M' \Delta G$ s.t G/E supersolvable. If every maximal of every $S_{y,l}(E)$ is Nearly Hall s-embedded then G is supersolvable.

Data Availability

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REFERENCE

- Asaad, M., & Heliel, A. A. (2001). On S-quasinormally embedded subgroups of finite groups. *Journal of Pure and Applied Algebra*, 165(2), 129-135.
- Z. Chen, on a theorem of srinivasan, J. *Southwest Normal Univ. Nat. sci.(chinese)*, 12(1) (1987), 1-4.
- W. E. Deskins, On quasinormal subgroups of finite groups, *Math. Z.*, 82(1963), 125-132.
- K. Doerk and T. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin, 1992.
- Y. Gao and X. Li, On NH-embedded subgroups of finite groups, *J. Algebra Appl.*, 21(10) (2022), 2250200 (11pp).
- B. Huppert, *Er dliche Gruppen*, Springer-Verlag, Berlin, 1967.
- Al-Sharo, K. A. (2012). On nearly S-permutable subgroups of finite groups. *Communications in Algebra*, 40(1), 315-326.
- Gao, Y., & Li, X. (2022). On NH-embedded subgroups of finite groups. *Journal of Algebra and Its Applications*, 21(10), 2250200.
- Deskins, W. E. (1963). On quasinormal subgroups of finite groups. *Mathematische Zeitschrift*, 82(2), 125-132.
- Schmid, P. (1998). Subgroups permutable with all Sylow subgroups. *Journal of Algebra*, 207(1), 285-293.
- Asaad, M., & Heliel, A. A. (2001). On S-quasinormally embedded subgroups of finite groups. *Journal of Pure and Applied Algebra*, 165(2), 129-135.
- Qinhai, Z., & Lifang, W. (2005). The influence of s-semipermutable subgroups on the structure of finite groups. *J. Math.*, 48(1), 81-88.
- Li, Y., Qiao, S., Su, N., & Wang, Y. (2012). On weakly s-semipermutable subgroups of finite groups. *Journal of Algebra*, 371, 250-261.
- Doerk, K., & Hawkes, T. (1992). *Finite Soluble Groups*, Walter de Gruyter. Berlin–New York, 889.